Plasma two-temperature equilibration rate

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A theory is developed that is suitable for describing a two-species thermalization process in a plasma with parameters suitable for recombination to take place. Recombining plasmas have recently been produced using positrons and antiprotons [M. Amoretti *et al.*, Nature (London) **419**, 456 (2002); G. Gabrielse *et al.*, Phys. Rev. Lett. **89**, 213401 (2002)]. The theory is not restricted to large Coulomb logarithm values, and correspondence with prior theory is shown in the limit of large Coulomb logarithm values. The theory applies for two plasma species, each having a Maxwellian velocity distribution and being weakly correlated.

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Nested Penning traps have now been used to attain overlapping confinement regions for positrons and antiprotons such that antihydrogen recombination has been achieved [1,2]. The attainment of overlapping confinement regions for two oppositely signed plasma species in a nested Penning trap appears to be possible even if each species has a Maxwellian velocity distribution [3]. However, the two species must be associated with disparate temperatures if each has a Maxwellian velocity distribution [3]. In this Brief Report, the thermalization of two weakly correlated, Maxwellian plasma species (e.g., one positron and one antiproton) is considered theoretically. An expression for calculating the two-temperature equilibration rate is developed that applies for any value of the Coulomb logarithm. Spitzer [4,5] formulated an equilibration rate for a two-component plasma based on Chandrasekhar's formulas for the velocity space friction and diffusion coefficients [6,7]. Due to the approximations employed, Spitzer's formula for the equilibration rate applies for large values ($\gtrsim 10$) of the Coulomb logarithm. However, weakly correlated, recombining plasmas may be associated with small Coulomb logarithm values [3,8].

Consider a single test particle moving in a plasma of Maxwellian field particles. The test particle will exchange its energy with the field particles due to collisions. In a single encounter of a test particle with a field particle, the exchange of energy ΔE is given by

$$\Delta E = \tfrac{1}{2} m (\Delta v^2 + 2 v \Delta v_\parallel), \tag{1}$$

where m is the mass of the test particle, v is its speed, Δv is the magnitude of its change in velocity due to the collision, and $\Delta v_\parallel$ is the magnitude of its change in velocity along its direction of motion before the collision. Averaging Eq. (1) over a Maxwellian velocity distribution for the field particles, the average time rate of change of the test particle energy is written as

$$\langle \Delta E \rangle = \tfrac{1}{2} m (\langle \Delta v^2 \rangle + 2 v \langle \Delta v_\parallel \rangle). \tag{2}$$

In Eq. (2), $\langle \Delta v_\parallel \rangle$ and $\langle \Delta v^2 \rangle$ are Fokker-Planck velocity-space friction and diffusion coefficients. Suppose that a group of test particles have a Maxwellian velocity distribution with temperature T, which is different from that of the field particles of temperature T_{F}. Averaging Eq. (2) over a Maxwellian velocity distribution for the test particles provides the rate of change of the test particle temperature,

$$\frac{3}{2} n k \frac{dT}{dt} = \int \langle \Delta E \rangle f_{\mathrm{M}}(v,T) d\mathbf{v}, \tag{3}$$

where n is the density of the test particle species, k is Boltzmann's constant, and $f_{\mathrm{M}}(v,T)$ is the Maxwellian velocity distribution with temperature T,

$$f_{\mathrm{M}}(v,T) = n \left(\frac{m}{2\pi k T} \right)^{3/2} \exp\left(-\frac{m}{2kT} v^2 \right). \tag{4}$$

Because the test particle velocity distribution is isotropic, an integral over solid angle can be carried out and Eq. (3) can be written as

$$\frac{dT}{dt} = \frac{8\pi}{3nk} \int_0^\infty \langle \Delta E \rangle f_{\mathrm{M}}(v,T) v^2 dv. \tag{5}$$

Substituting Eq. (4) into Eq. (5) and carrying out a variable change, $u = v/v_{\mathrm{th}}$ with the thermal velocity of a field particle defined as $v_{\mathrm{th}} = \sqrt{2kT_{\mathrm{F}}/m_{\mathrm{F}}}$, a nondimensional form of Eq. (5) can be written as

$$\frac{dT}{dt} = \frac{8 \zeta^{3/2}}{3\sqrt{\pi} k} \int_0^\infty \langle \Delta E \rangle \exp(-\zeta u^2) u^2 du. \tag{6}$$

Here, $\zeta = T_{\mathrm{F}} m / (T m_{\mathrm{F}})$, and m_{F} is the mass of a field particle. In order to integrate Eq. (6), expressions for friction and diffusion coefficients must be substituted into Eq. (2).

Spitzer employed Chandrasekhar's expressions for the friction and diffusion coefficients [5],

$$\langle \Delta v_\parallel \rangle = -\frac{4 a v_{\mathrm{th}} \lambda G(u)}{\tau_0}, \tag{7}$$

and

$$\langle \Delta v^2 \rangle = \frac{2 (a v_{\mathrm{th}})^2 \lambda \, \mathrm{erf}(u)}{\tau_0 u}, \tag{8}$$

where $a = 2\mu/m$, the reduced mass is $\mu = m m_{\mathrm{F}}/(m + m_{\mathrm{F}})$, λ is the Coulomb logarithm, the time parameter for single particle interactions is $\tau_0=(n_F v_{th}\pi r_0^2)^{-1}$, n_F is the density of the field particles, the interaction radius is defined as $r_0 = ZZ_F e^2/[8\pi\epsilon_0 k(\mu T_F/m_F)]$, Z and Z_F are the charge state of a test particle and field particle, e is the unit charge, ϵ_0 is the permittivity of free space, erf is the error function, and Chandrasekhar's function is defined as $G(u)= -\frac{1}{2}d/du[\mathrm{erf}(u)/u]$. Substituting Eqs. (7) and (8) into Eq. (2), the integral in Eq. (6) can be carried out as

$$\frac{dT}{dt}=\frac{4m(av_{th})^2\lambda}{3\sqrt{\pi}\tau_0 k}\frac{\zeta^{1/2}(1+\zeta-2a^{-1})}{(1+\zeta)^{3/2}}, \tag{9}$$

where the Coulomb logarithm has been approximated as being constant. Equation (9) can be rearranged into the form,

$$\frac{dT}{dt}=\frac{T_F-T}{\tau_{eq}^{\mathrm{Spitzer}}}, \tag{10}$$

where $\tau_{eq}^{\mathrm{Spitzer}}$ is given by

$$\tau_{eq}^{\mathrm{Spitzer}}=\frac{3mm_F}{8\sqrt{2\pi}n_F\lambda}\left(\frac{4\pi\epsilon_0}{ZZ_F e^2}\right)^2\left(\frac{kT}{m}+\frac{kT_F}{m_F}\right)^{3/2}. \tag{11}$$

Spitzer's equilibration rate, given by Eqs. (10) and (11), applies when the Coulomb logarithm is large both because it was approximated as constant and because an approximation was employed in the process of obtaining Chandrasekhar's expressions for the friction and diffusion coefficients.

A variable change technique [8–13] has been developed, which has been used for deriving Fokker-Planck coefficients. Exact onefold integral expressions for Fokker-Planck velocity-space friction and diffusion coefficients have been obtained using the technique [8]. These expressions make it possible to recalculate the equilibration rate without placing a restriction on the value of the Coulomb logarithm. The expressions for the friction and diffusion coefficients are [8]

$$\langle\Delta v_\parallel\rangle=-\frac{av_{th}}{\tau_0 u^2}\int\left(\frac{\mathrm{erf}(U)+\mathrm{erf}(W)}{u_\delta}+\frac{\exp(-U^2)-\exp(-W^2)}{\sqrt{\pi}u_\delta^2}\right)du_\delta \tag{12}$$

and

$$\langle\Delta v^2\rangle=\frac{(av_{th})^2}{\tau_0 u}\int\frac{\mathrm{erf}(U)+\mathrm{erf}(W)}{u_\delta}du_\delta, \tag{13}$$

where $U=u+u_\delta$, $W=u-u_\delta$, and $u_\delta=\Delta v/(av_{th})$ is a nondimensional variable for change in velocity. Substituting Eqs. (12) and (13) into Eq. (2), Eq. (6) can be expressed as the following twofold integral:

$$\frac{dT}{dt}=\frac{4m(av_{th})^2\zeta^{3/2}}{3\sqrt{\pi}\tau_0 k}\int\int\left(\frac{\mathrm{erf}(U)+\mathrm{erf}(W)}{(2a^{-1}-1)^{-1}u_\delta}-\frac{\exp(-U^2)-\exp(-W^2)}{2^{-1}\sqrt{\pi}au_\delta^2}\right)\frac{udu\,du_\delta}{\exp(\zeta u^2)}. \tag{14}$$

The integral in Eq. (14) over u can be evaluated to obtain

$$\frac{dT}{dt}=\frac{4m(av_{th})^2}{3\sqrt{\pi}\tau_0 k}\frac{\zeta^{1/2}(1+\zeta-2a^{-1})}{(1+\zeta)^{3/2}}\times\int\exp\left(-\frac{\zeta u_\delta^2}{1+\zeta}\right)u_\delta^{-1}du_\delta, \tag{15}$$

or

$$\frac{dT}{dt}=\frac{T_F-T}{\tau_{eq}}, \tag{16}$$

and

$$\tau_{eq}=\frac{3mm_F\tau}{16\mu^2}\left(\frac{1}{2}\int\exp(-x)x^{-1}dx\right)^{-1}. \tag{17}$$

Here, $\tau=(n_F v_{avg}\pi r^2)^{-1}$, where the average relative speed between two different groups of Maxwellian particles is $v_{avg}=\sqrt{8kT'/(\pi\mu)}$, and the interaction radius is defined as $r=ZZ_F e^2/(8\pi\epsilon_0 kT')$. The nondimensional interaction strength is defined as $x=H/(kT')$, the collision strength is defined as

$$H=\frac{\Delta p^2}{8\mu}, \tag{18}$$

the magnitude of the momentum transfer for a collision event is $\Delta p=m\Delta v$, and the reduced temperature is defined as

$$T'=\mu\left(\frac{T}{m}+\frac{T_F}{m_F}\right). \tag{19}$$

No approximations are used to arrive at Eqs. (16) and (17). However, the integral in Eq. (17) diverges due to the long-range Coulomb interaction, and a cutoff must be imposed on the nondimensional collision strength x. The minimum and maximum nondimensional change in velocity can be expressed in terms of the average relative speed and the Coulomb logarithm as [8]

$$u_{\delta,\min}=\frac{v_{avg}}{v_{th}}e^{-\lambda}, \tag{20}$$

$$u_{\delta,\max}=\frac{v_{avg}}{v_{th}}. \tag{21}$$

For the two-temperature system, the average relative speed is $v_{avg}=2[(1+\zeta)/(\pi\zeta)]^{1/2}v_{th}$, and the integral limits for Eq. (17) become

$$x_{\min}=\frac{4}{\pi}e^{-2\lambda}, \tag{22}$$

$$x_{\max}=\frac{4}{\pi}. \tag{23}$$

Substituting these into Eq. (17) gives

$$\tau_{\mathrm{eq}}=\frac{3mm_{\mathrm{F}}}{4\sqrt{2\pi}n_{\mathrm{F}}\Gamma\left(0,\frac{4}{\pi}e^{-2\lambda},\frac{4}{\pi}\right)}\left(\frac{4\pi\epsilon_0}{ZZ_{\mathrm{F}}e^2}\right)^2\left(\frac{kT}{m}+\frac{kT_{\mathrm{F}}}{m_{\mathrm{F}}}\right)^{3/2}, \tag{24}$$

where the zeroth-order incomplete gamma function $\Gamma(0,x_{\min},x_{\max})$ is defined as

$$\Gamma(0,x_{\min},x_{\max})=\int_{x_{\min}}^{x_{\max}}e^{-x}x^{-1}dx. \tag{25}$$

The difference between Spitzer's theory and the present theory is characterized by the ratio

$$\frac{\tau_{\mathrm{eq}}^{\mathrm{Spitzer}}}{\tau_{\mathrm{eq}}}=\frac{\Gamma\left(0,\frac{4}{\pi}e^{-2\lambda},\frac{4}{\pi}\right)}{2\lambda}, \tag{26}$$

which is only a function of the Coulomb logarithm. This ratio is shown in Fig. 1. The plot in Fig. 1 indicates that values calculated using Eq. (24) can be up to 3.6 times larger than those calculated using Eq. (11). When the Coulomb logarithm is larger than ten, however, the difference is less than 5%. Thus correspondence occurs for large Coulomb logarithm values, as expected.

An expression for the Coulomb logarithm that is consistent with the present theory is [8]

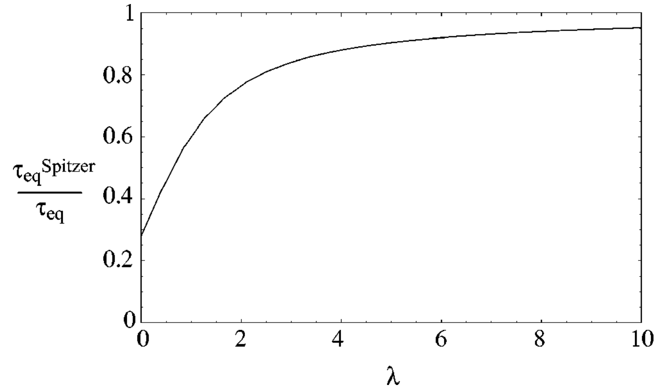

FIG. 1. A plot of $\tau_{\mathrm{eq}}^{\mathrm{Spitzer}}/\tau_{\mathrm{eq}}$ vs the Coulomb logarithm λ.

$$\lambda=\ln(\sqrt{1+\Lambda^2}), \tag{27}$$

where $\Lambda=\rho_{\max}/\beta$, $\rho_{\max}$ is the maximum impact parameter, and $\beta=ZZ_{\mathrm{F}}e^2/[32\epsilon_0\mu k(T/m+T_{\mathrm{F}}/m_{\mathrm{F}})]$ is the impact parameter for 90° scattering in the center of mass frame of reference as a result of a Coulomb collision. The maximum impact parameter that should be used for considering interactions between oppositely signed plasma species in Penning traps has recently come into question and is the subject of further investigation. Preliminary results indicate that the larger of the two cyclotron radii associated with the two plasma species should be used. Or, if the Debye length associated with the binary interactions is smaller than either cyclotron radius, then the Debye length should be used.

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